Introduction to Tensor Ranks and Tensor Invariants

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Tensor Ranks and Tensor Invariants Seminar — April 11th 2024

Outline

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What is a tensor?

Outer product

Tensor basis

Restriction

Outline

What is a tensor?		**Tensors in the wild**
Outer product	→	Matrix multiplication
Tensor basis		Quantum entanglement
Restriction		Combinatorics

Outline

What is a tensor?		**Tensors in the wild**		**Group actions on tensors**
Outer product	$\rightarrow$	Matrix multiplication	$\rightarrow$	Diagonal action
Tensor basis		Quantum entanglement		Permutation action
Restriction		Combinatorics		(Anti)symmetric tensors

Tensors as multidimensional arrays

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Starting point — Matrices

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for vectors $v_i \in \mathbb{F}^n$, $w_i \in \mathbb{F}^m$.

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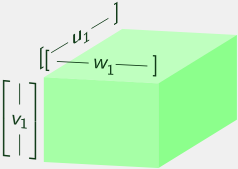
$T =$ 

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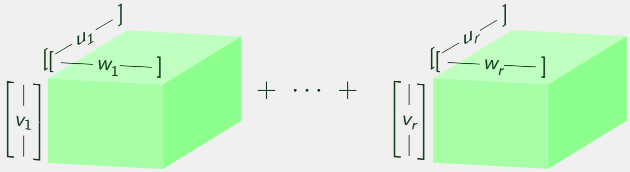

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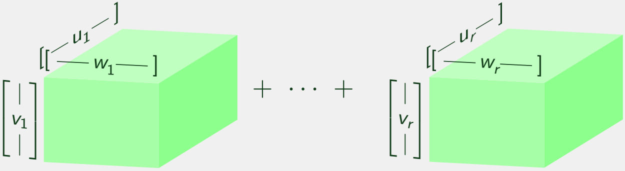

First examples — The W and diagonal tensors

Tensors as multidimensional arrays

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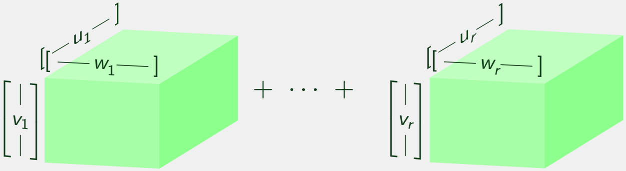

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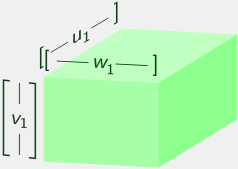
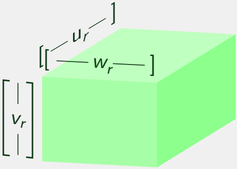


$+ \cdots +$



$$W := e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$$

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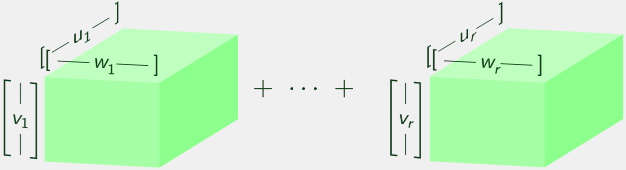

$$
\begin{aligned}
W &:= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \\
&= \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}
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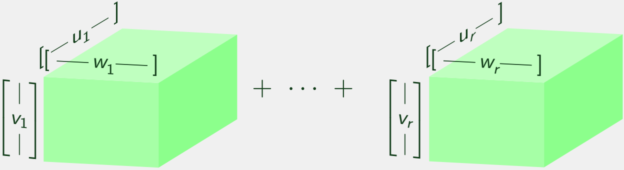

$$\begin{aligned} W \quad &:= \quad e_1 \otimes e_1 \otimes e_2 \quad + \quad e_1 \otimes e_2 \otimes e_1 \quad + \quad e_2 \otimes e_1 \otimes e_1 \\ &= \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad + \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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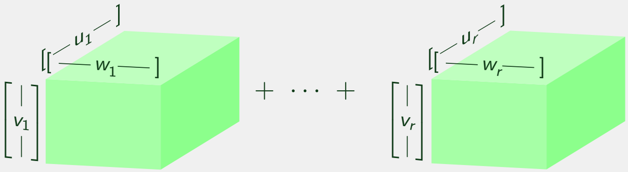

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W &:= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \\
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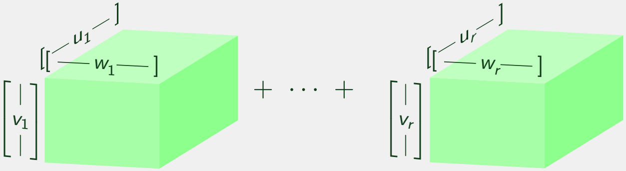

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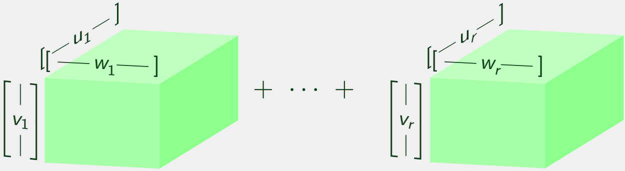

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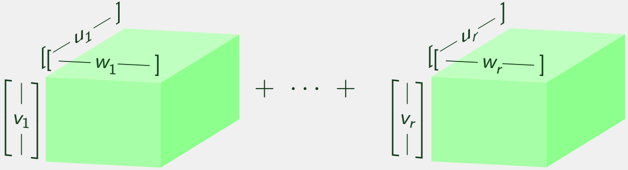

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$$\langle 2 \rangle := e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right)$$

Abstract tensors

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Definition — Abstract 3-tensor space *(Straightforward to generalize to k-tensors)*

We define a **tensor vector space** $V \otimes W \otimes U$ as the linear span of the (abstract) elements

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It is easy to check the outer product satisfies this!

Kronecker product

You could also define:

Another example — Kronecker product

Given column vectors $v \in V$, $w \in W$. Define their **Kronecker product** by

$$v \boxtimes w = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \boxtimes w := \begin{bmatrix} a_1 w \\ \hline a_2 w \\ \hline \vdots \\ \hline a_n w \end{bmatrix} \in V \boxtimes W$$

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i.e. replacing each entry of v with a scaled copy of w, resulting in one very tall vector.

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This also sasisfies the abstract definition!

How to transform tensors — Linear operations

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Remark: Restriction on matrices (2-tensors) is left-right multiplication, since

$$(A \otimes B)(v \otimes w) = Av \otimes Bw = Av(Bw)^\top = A(vw^\top)B^\top.$$

Matrix multiplication and Bilinear maps

$$\mathrm{MM}_n \colon \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$$

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- Then we write $f(v_i, w_i) = \sum_k \left(\underbrace{u_k^* f(v_i, w_i)} \right) u_k$

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Matrix multiplication and Bilinear maps

$$\mathrm{MM}_n \colon \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$$

Standard algorithm: $\mathcal{O}(n^3)$. **Best bounds:** $\mathcal{O}(n^\omega)$ with $\omega \in [2,\ 2.371552]$.

Central question — Matrix multiplication

How many multiplications (between inputs) are needed to do $n \times n$ matrix multiplication?

Consider bilinear maps $V \times W \to U$, with $\{v_i\}_i$, $\{w_j\}_j$ and $\{u_k\}_k$ bases. **Claim:**

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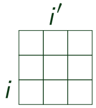

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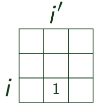

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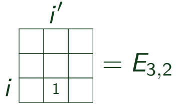

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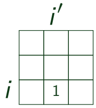

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This is just the beginning of the story. In this seminar we will/might see:

- A session on tensor rank
- A session on border bank
- Asymptotic aspects
- *Student topic:* Schönhage's τ-theorem

Quantum states

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A **qubit** is the system $\mathbb{C}^2$. Examples of states: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $|+\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (superposition).

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A **qubit** is the system $\mathbb{C}^2$. Examples of states: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $|+\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (superposition).
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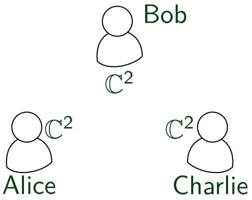

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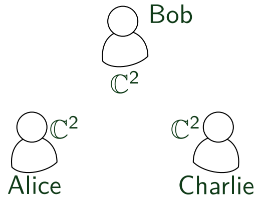

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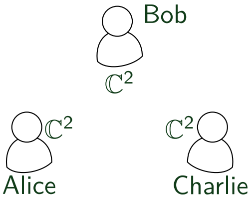

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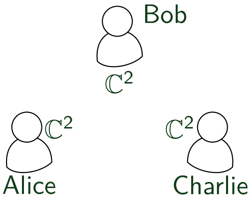

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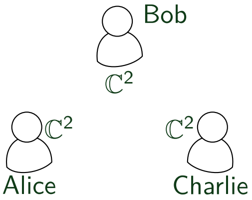

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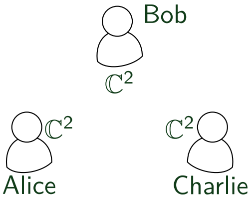

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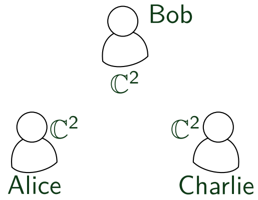

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$$\mathrm{R}_1(\mathrm{W}) = \mathrm{rank}\big(e_1 \otimes (e_1 \boxtimes e_2) + e_1 \otimes (e_2 \boxtimes e_1) + e_2 \otimes (e_1 \boxtimes e_1)\big)$$

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Proof: Restriction $(A \otimes B \otimes C)T$ becomes left-right matrix multiplication $(A)M_T(B \boxtimes C)^*$. □

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We say a function $f\colon \{\text{3-tensors}\} \to \mathbb{R}$ is **monotone** when $S \leq T \implies f(S) \leq f(T)$.

Theorem — Flattening ranks

Given $T \in V \otimes W \otimes U$ we can consider T as a matrix $M_T \in V \otimes (W \otimes U)$, and compute matrix rank. We call this the 1st **flattening rank** R_1. Then $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ are restriction monotones.

Proof: *Restriction* $(A \otimes B \otimes C)T$ *becomes left-right matrix multiplication* $(A)M_T(B \boxtimes C)^*$. $\square$

$$\mathrm{R}_1(\mathrm{W}) = \operatorname{rank}\big(e_1 \otimes (e_1 \boxtimes e_2) + e_1 \otimes (e_2 \boxtimes e_1) + e_2 \otimes (e_1 \boxtimes e_1)\big)$$

$$= \operatorname{rank}\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right) = \operatorname{rank}\begin{bmatrix}0&1&1&0\\1&0&0&0\end{bmatrix} = 2$$

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There exists an semi-invariant f for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with $f(\langle 2\rangle) \neq 0 = f(\mathrm{W})$. It is called the **hyperdeterminant** or **3-tangle**.

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Proof. We might see this as part of a student topic :)

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- Thus: $\langle 2 \rangle / \sqrt{2}$ has a genuinly different type of entanglement than $\mathrm{W}/\sqrt{3}$.

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Again just the beginning of the story. In this seminar we will/might see:

- Schur–Weyl duality, covariants
- The quantum functionals
- More monotones, (semi-)invariants
- *Student topic:* classification of classes in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

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- We can reformulate this result in terms of tensors!

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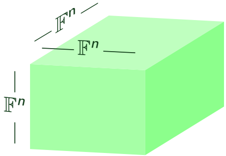

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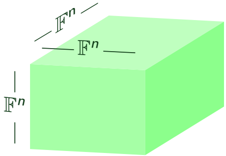

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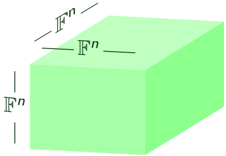

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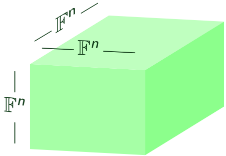

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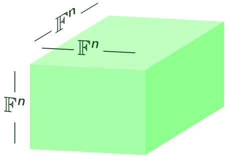

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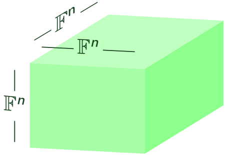

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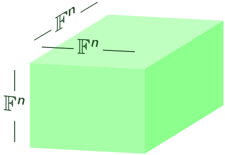

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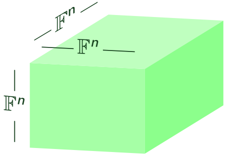

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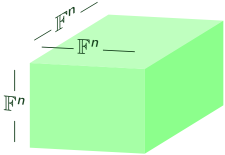

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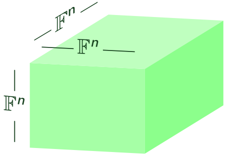

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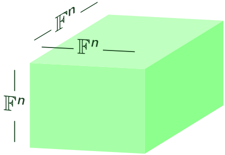

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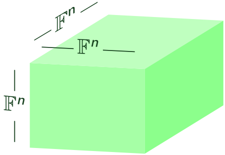

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Once again again the beginning of the story. In this seminar we will/might see:

- A session on subrank
- More upper bounds for subrank
- A general asymptotic formulation
- *Student topic:* slice rank

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The tensor world is a lot more complicated & interesting than the matrix world!
We use ranks (rank, subrank, slice rank, ...), monotones, invariants, etc.

Group actions

Group actions

Recall the definition of invariants.

Definition — Restriction semi-invariant

We say a function $f\colon V \otimes W \otimes U \to \mathbb{R}$ is an **semi-invariant** when $f\big((A \otimes B \otimes C)T\big) = 0 \iff f(T) = 0$ for all invertible $(A, B, C) \in \mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{GL}(U)$.

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Examples:

$$v \otimes \cdots \otimes v$$

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- The set of antisymmetric tensors in $V^{\otimes n}$ form a vector space.
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- $v_1 \wedge \cdots \wedge v_n = 0 \iff \{v_1, \ldots, v_n\}$ are linearly dependent. *(hint: consider first $v_i = v_j$)*

Slides will be available at the webpage: *qi.rub.de/tensors_ss24*.

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That's it for today. Thanks!